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# SPACE-TIME FINITE ELEMENT FORMULATION FOR SHALLOW WATER EQUATIONS WITH SHOCK-CAPTURING OPERATOR

**Rigoberto G. Sanabria Castro**
e-mail: sanabria@lncc.br

UNIVERSIDAD NACIONAL MAYOR DE SAN MARCOS

FACULTAD DE CIENCIAS MATEMÁTICAS

ABSTRACT This paper presents a space-time formulation for problems governed by the shallow water equations. A linear time-discontinuous approximation is adopted and the streamline upwind Petrov-Galerkin (SUPG) method is applied in its equivalent form to fit the time discretization. Also, a shock-capturing operator is used in order to solve all details of sharp layers and/or shock discontinuities. The semi-discrete version is also established and numerical examples compare the performance of these methods.

## 1. INTRODUCTION

The space-time Petrov-Galerkin ($STPG$) method for the solution of shallow water equations has been presented in [1] . This method is based on the time-discontinuous Galerkin formulation with the addition of the Petrov-Galerkin operator.

The Galerkin and the time-discontinuous Galerkin methods lack stability in the approximation of convection dominated phenomena. A manifestation of this lack of stability is that spurious oscillations spread over the entire computational domain, generated by unresolved internal and boundary layers. The operator added in the $STPG$ method controls the derivatives along the characteristics, resulting in good stability and accuracy properties and shows a convergence improvement over the time-discontinuous Galerkin method. However, in the neighborhood of regions containing sharp gradients the approximate solution may exhibit over- and under-shoots.

In this paper we present two Shock-Capturing ($SC$) operators, both leading to stable and accurate method which are capable of solving all details of sharp layers and/or shock discontinuities. The first operator is the so-called $CAU$ proposed in[2, 3] , which is an extension of the operator proposed by Galeaõ and Dutra do Carmo[4] for the scalar advective-diffusive equation. The second operator has been developed by Shakib[5] for the Navier-Stokes equations. For the space-time formulation, linear space-continuous and time-discontinuous approximations are used.

## 2. PROBLEM STATEMENT

Let $(x, y) \in \Omega \subset \mathbb{R}^2$ define a set of points on an horizontal plane and let $z \in [h, \eta]$ denote the vertical direction, where $h(x, y)$ represents the water depth and $\eta(x, y, z)$ is the water surface elevation, both measured from the undisturbed water surface. We start from the $3-D$ incompressible Navier-Stokes equations, after turbulent time-averaging, integrating these equations along the $z$ direction using depth-averaged horizontal velocities. Under the simplifying assumption of a hydrostatic pressure distribution (negligible vertical acceleration), we arrive at the shallow water equations:

$$
\begin{aligned}
u_{,t} + uu_{,x} + vu_{,y} + (gH)_{,x} - gh_{,x} - fv - \sigma w^x - ru + \mu(u_{,xx} + u_{,yy}) &= 0 \\
v_{,t} + uv_{,x} + vv_{,y} + (gH)_{,y} - gh_{,y} + fu - \sigma w^y - rv + \mu(v_{,xx} + v_{,yy}) &= 0
\end{aligned}
$$

$$
H_{,t} + Hu_{,x} + Hv_{,y} + uH_{,x} + vH_{,y} = 0 \tag{2.1}
$$

In these equations, $H = h + \eta$ is the total water depth, $u$ and $v$ are the averaged components of the velocity in $x$ and $y$ directions respectively. The gravitational acceleration is given by $g$ and $f$ is the Coriolis parameter, $\sigma$ is the surface friction coefficient and $w^x$, $w^y$ are the wind velocity components, $\mu$ is the eddy viscosity.

Multiplying the third equation by $g$ and observing that,

$$
(gH)_{,t} = \left(\left(\sqrt{gH}\right)^2\right)_{,t} = \sqrt{gH}\left(2\sqrt{gH}\right)_{,t} = c\,(2c)_{,t} \tag{2.2}
$$

where $c = (gH)^{1/2}$ , and considering similar expressions for $(gH)_{,x}$ and $(gH)_{,y}$ we obtain the shallow water equations in the velocity-celerity variables [6] which, in matrix form, can be written as:

$$
U_{,t} + \mathcal{A}.\nabla U - \nabla.(\mathbb{K}\nabla U) + CU = \mathcal{F} \tag{2.3}
$$

where:

$$
U^T = \begin{bmatrix} u & v & \theta \end{bmatrix}; \qquad \mathcal{A}^T = \begin{bmatrix} A_1^T & A_2^T \end{bmatrix}; \qquad \theta = 2c \tag{2.4}
$$

$$
A_1 = \begin{bmatrix} u & 0 & c \\ 0 & u & 0 \\ c & 0 & u \end{bmatrix}; \qquad A_2 = \begin{bmatrix} v & 0 & 0 \\ 0 & v & c \\ 0 & c & v \end{bmatrix}; \qquad C = \begin{bmatrix} r & -f & 0 \\ f & r & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.5}
$$

(2.6) $$\mathbb{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}; \qquad K_{11} = K_{22} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2.7) $$K_{12} = K_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \qquad \mathcal{F} = \begin{bmatrix} gh_{,x} + \sigma w^x \\ gh_{,y} + \sigma w^y \\ 0 \end{bmatrix}$$

(2.8) $$\mathcal{A}.\nabla U = \mathcal{A}^T \nabla U; \qquad \nabla U = \begin{bmatrix} I_3 \partial/\partial x \\ I_3 \partial/\partial y \end{bmatrix} U$$

In (2.3) $\mathcal{A}.\nabla U$ plays the role of a generalized advection term and $\nabla.(\mathbb{K}\nabla U)$ plays the role of a generalized diffusion operator.

Once an initial state $U_o(x,)$ is specified at $t = 0$ and appropriate boundary conditions are prescribed, the system of equations above can be solved to give the unknown column vector $U$.

## 3. STPG WITH SHOCK-CAPTURING FINITE ELEMENT MODEL

In order to construct the space-time finite element subspace, let us consider $\Omega$ a bounded open set of $\mathbb{R}^2$ with boundary $\Gamma$. Let us also consider the partition $0 = t_0 < t_1 < ... < t_N = T$ of the interval $I = (0, T)$ and denote by $I_n = (t_n, t_{n+1})$ the $n^{th}$ time interval. For each $n$ the space-time integration domain is the "slab" $S_n = \Omega \times I_n$, with boundary $\overline{\Gamma}_n = \Gamma \times I_n$; see Figure 3.1.

If we define $S_n^e$ as the $e^{th}$ element in $S_n$, $e = 1, 2, .., (N_e)_n$, where $(N_e)_n$ is the total number of elements in $S_n$, then for $n = 0, 1, 2, ..$ we have:

($i$) The space-time finite element partition $\Pi^{h,\Delta t}$ is such that:

(3.1) $$S_n = \cup_{e=1}^{(nel)_n} \overline{S}_n^e \quad , \qquad S_n^e = \Omega_n^e \times I_n$$

(3.2) $$\Omega = \cup_{e=1}^{(nel)_n} \overline{\Omega}_n^e \quad , \qquad \Omega_n^i \cap \Omega_n^j = \emptyset \qquad i \neq j$$

($ii$) The space-time finite element subspace consists of continuous piecewise polynomials on the "slab" $S_n$, and may be discontinuous in time across the time levels $t_n$, that is:

*trial functions* :

(3.3) $$\mathcal{U}_n^h = \left\{ U^h \;\; ; \;\; U^h \in (C^o(S_n))^3 ; \;\; U^h|_{S_n^e} \in (\mathbb{P}_k(S_n^e))^3 ; \;\; U^h|_{\overline{\Gamma}_n} = g \right\}$$

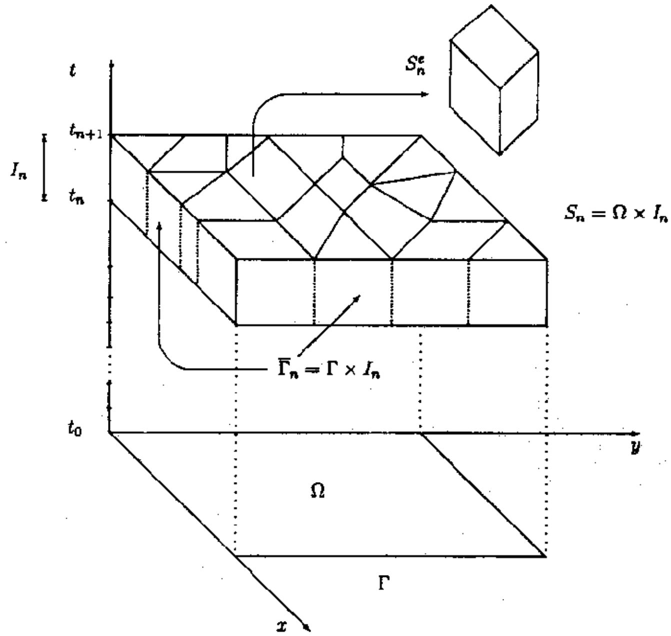


Figure 3.1. Slab space-time.

*weighting functions* :

$$(3.4) \qquad \widehat{\mathcal{U}}_n^h = \left\{ \widehat{U}^h \ ; \ \widehat{U}^h \in (C^o(S_n))^3 ; \ \widehat{U}^h|_{S_n^e} \in (\mathbb{P}_k(S_n^e))^3 ; \ \widehat{U}^h|_{\overline{\Gamma}_n} = 0 \right\}$$

where $\mathbb{P}_k$ is the set of polynomials of degree less than or equal to $k$ and $g$ are the prescribed boundary conditions.

Considering that finite element functions are discontinuous at the space-time slab interfaces, let

$$(3.5) \qquad \widehat{U}^h(t_n^{\pm}) = \lim_{\epsilon \to 0^{\pm}} \widehat{U}^h(t_n + \epsilon)$$

and define the *jump* in time $t_n$ of $\widehat{U}^h$ as

$$(3.6) \qquad [\![\widehat{U}^h(t_n)]\!] = \widehat{U}^h(t_n^+) - \widehat{U}^h(t_n^-)$$

According to the above definitions, the variational space-time formulation discontinuous in time for the problem (2.3) reads:

Within each $S_n$, $n = 0, 1, 2, ..$, find $U^h \in \mathcal{U}_n^h$ such that for all $\widehat{U}^h \in \widehat{\mathcal{U}}_n^h$ the following variational equation is satisfied

$$(3.7)\quad \int_{S_n} \widehat{U}^h.\mathcal{R}U^h d\Omega dt + \int_{\Omega} \widehat{U}^h(t_n^+).[\![U^h(t_n)]\!] d\Omega$$
$$+ \sum_{e=1}^{(N_e)_n} \int_{S_n^e} \tau\left(\widehat{U}^h_{,t} + \mathcal{A}.\nabla\widehat{U}^h\right).\mathcal{R}U^h d\Omega dt$$
$$+ \sum_{e=1}^{(N_e)_n} \int_{S_n^e} \tau_c\, \widehat{\nabla}_\xi \widehat{U}^h.\widehat{\nabla}_\xi U^h d\Omega dt = 0$$

where

$$(3.8)\quad \mathcal{R}U^h = U^h_{,t} + \mathcal{A}.\nabla U^h - \nabla.\left(I\!\!K\nabla U^h\right) + \mathcal{C}U^h - \mathcal{F} = \mathcal{L}U^h - \mathcal{F}$$

is the residual. $\mathcal{L}$ is defined as

$$(3.9)\qquad \mathcal{L} = \frac{\partial}{\partial t} + A_i\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i}(K_{ij}\frac{\partial}{\partial x_j}) + C$$

The first, second and last integrals in (3.7) constitute the time - discontinuous Galerkin formulation. The jump condition is the mechanism by which the information is propagated from one space-time slab to the next. The third integral in (3.7) is the $(STPG)$ operator and the fourth integral is the shock-capturing operator $(SC)$. Those operators will be briefly seen later.

**3.1 Space-Time Petrov-Galerkin Operator (STPG).** The Space-Time Petrov-Galerkin operator is defined as

$$(3.10)\qquad \sum_{e=1}^{(N_e)_n} \int_{S_n^e} \tau\left(\widehat{U}^h_{,t} + \mathcal{A}.\nabla\widehat{U}^h\right).\mathcal{R}U^h d\Omega dt$$

where $\tau$ is the $3\times 3$ symmetric positive-semidefinite Petrov-Galerkin matrix of intrinsic time scales. The definition of this matrix $\tau$ is given in[5] by

$$(3.11)\quad \tau = \left(\left(\frac{\partial\xi_0}{\partial x_0}\right)^2 I_3 + \left(\frac{\partial\xi_i}{\partial x_j}\frac{\partial\xi_i}{\partial x_k}\right)\mathcal{A}_j\mathcal{A}_k\right.$$
$$\left. + \left(\frac{\partial\xi_i}{\partial x_k}\frac{\partial\xi_j}{\partial x_l}\frac{\partial\xi_j}{\partial x_m}\frac{\partial\xi_i}{\partial x_n}\right)K_{kl}K_{mn} + \mathcal{C}^2\right)^{-1/2}$$

where $x_o = t : x_1 = x$ ; $x_2 = y$ ; $\xi_k(k = 0, 1, 2)$ are the local coordinates of the parent element $S_n^e$ and $I_3$ is the identity matrix of dimension 3. When solving steady state problems or if a semi-discrete formulation is used, the first term in the square-root inverse (3.11) may be set to zero.

**3.2 Shock-Capturing Operator.** This operator is built to satisfy a few design conditions: in order to control the oscillations, this operator should act in the direction of the gradient; for consistency it should be proportional to the residual $\mathcal{R}U^h$; and for accuracy it should vanish quickly in regions where the solution is smooth.

We present two operators that satisfy the above conditions.

3.2.1 *CAU Operator.* This operator is defined as [2, 3]

$$\sum_{e=1}^{(N_e)_n} \int_{S_n^e} \tau_c \, \nabla \widehat{U}^h . \nabla U^h d\Omega dt \tag{3.12}$$

where

$$\tau_c = \max \left\{ 0, \zeta(P_e)\mu_c^{-1} \frac{|\mathcal{R}U^h|^2}{|\nabla U^h|^2} - \frac{\left(U^h_{,t} + \mathcal{A}.\nabla U^h\right)^T \tau \mathcal{R}U^h}{|\nabla U^h|^2} \right\} \tag{3.13}$$

$\zeta(P_e) = \coth(P_e) - \frac{1}{P_e}$ ; $P_e$ is the Peclet number[7] and

$$\mu_c^2 = \begin{cases} \frac{|\mathcal{R}U^h|^2 |\nabla_\xi U^h|^2}{|\nabla U^h|^4} & \text{if} \quad |\nabla U^h|^2 \neq 0 \\ 0 & \text{if} \quad |\nabla U^h|^2 = 0 \end{cases} \tag{3.14}$$

with

$$\nabla_\xi U^h = \begin{bmatrix} \frac{\partial}{\partial \xi_1} I_3 \\ \frac{\partial}{\partial \xi_2} I_3 \end{bmatrix} U^h \tag{3.12}$$

3.2.2 *DC Operator.* In order to define this operator, denote by $\nabla_\xi$, $\nabla_{\xi'}$ the local gradient in element spatial and space-time coordinate systems, respectively. That is,

$$\nabla_\xi = \begin{bmatrix} \frac{\partial}{\partial \xi_1} I_3 \\ \frac{\partial}{\partial \xi_2} I_3 \end{bmatrix} \qquad 6 \times 3 \tag{3.16}$$

$$\nabla_{\xi'} = \begin{bmatrix} \frac{\partial}{\partial \xi_0} I_3 \\ \frac{\partial}{\partial \xi_1} I_3 \\ \frac{\partial}{\partial \xi_2} I_3 \\ \frac{\partial}{\partial \xi_3} I_3 \end{bmatrix} \qquad 9 \times 3 \tag{3.17}$$

$$\nabla^2_\xi = \begin{bmatrix} \frac{\partial^2}{\partial \xi_1^2} I_3 \\ \frac{\partial^2}{\partial \xi_1 \partial \xi_2} I_3 \\ \vdots \\ \frac{\partial^2}{\partial \xi_3^2} I_3 \end{bmatrix} \qquad 12 \times 3 \tag{3.18}$$

Let the generalized gradient operator in the local element coordinates be defined as

$$\widehat{\nabla}_\xi = \begin{bmatrix} I_3 \\ \nabla_\xi \\ \nabla^2_\xi \end{bmatrix} \qquad 24 \times 3 \tag{3.19}$$

The number of terms included in $\widehat{\nabla}_\xi$ depends on the differential operator $\mathcal{L}$. The above definition accounts for all the gradient terms in (3.9).

Therefore the operator (DC) is defined as

$$\sum_{e=1}^{(N_e)_n} \int_{S_n^e} \tau_c \, \widehat{\nabla}_\xi \widehat{U}^h . \widehat{\nabla}_\xi U^h d\Omega dt \tag{3.20}$$

where

$$\tau_c = \begin{cases} \frac{|\mathcal{R}U^h|_\tau}{|\nabla_\xi U^h|_\tau} & \textit{Linear Form} \\ 2\frac{|\mathcal{R}U^h|^2_\tau}{|\nabla_\xi U^h|^2} & \textit{Quadratic Form} \end{cases} \tag{3.21}$$

$|\mathcal{R}U^h|^2_\tau = \mathcal{R}U^h . \tau \mathcal{R}U^h$

## 4. COMPUTATIONAL ASPECTS

**4.1. Linear-in-time and linear-in-space approximation.** In this approximation the $U^h$ and $\widehat{U}^h$ interpolations are bilinear in space and time. The total number of nodal points for each space-time slab is $2n_{np}$: $n_{np}$ nodal points at $t^-_{n+1}$ with values denoted by $U^h_{j;(n+1)^-}$ and $n_{np}$ nodal points at $t^+_n$ with values denoted by $U^h_{j;(n)^+}$, where $n_{np}$ is the number of spatial nodal points; (see Figure 4.1).

We refer to $U^h_{j;(n+1)^-}$ as the primary variables and to $U^h_{j;(n)^+}$ as the secondary variables. With these definitions, for the $n^{th}$ space-time slab we have

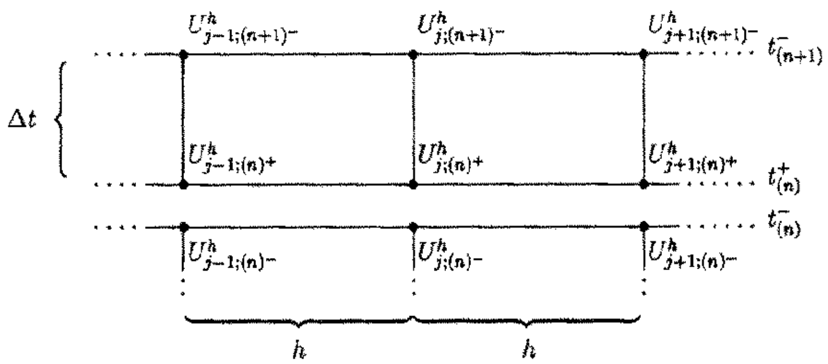


Figure 4.1. Nodal configuration for the linear-in-time approximation.

$$U^h(x,y,t) = \sum_{j=1}^{n_{np}} \varphi_j(x,y)\left(N_1(t)U^h_{j;(n)^+} + N_2(t)U^h_{j;(n+1)^-}\right) \tag{4.1}$$

$$\widehat{U}^h(x,y,t) = \sum_{j=1}^{n_{np}} \varphi_j(x,y)\left(N_1(t)\widehat{U}^h_{j;(n)^+} + N_2(t)\widehat{U}^h_{j;(n+1)^-}\right) \tag{4.2}$$

where $\varphi_j(x,y)$ is the spatial shape-function of spatial node $j$ which is assumed to be piecewise linear; $N_1(t)$ and $N_2(t)$ are the temporal shape-functions defined as

$$N_1(t) = \frac{t_{n+1} - t}{\Delta t} \tag{4.3}$$

$$N_2(t) = \frac{t - t_n}{\Delta t} \tag{4.4}$$

Substitution at the above functions into the variational equation (3.7) leads to following systems of equations:

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{bmatrix} U_{(n)^+} \\ U_{(n+1)^-} \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix} \tag{4.5}$$

where

$$K^{11} = K^{11}_G + K^{11}_{PG} + K^{11}_{SC} + M^{11}_t + M \tag{4.6}$$

$$K^{12} = K_G^{12} + K_{PG}^{12} + K_{SC}^{12} + M_t^{12} \tag{4.7}$$

$$K^{21} = K_G^{21} + K_{PG}^{21} + K_{SC}^{21} + M_t^{21} \tag{4.8}$$

$$K^{22} = K_G^{22} + K_{PG}^{22} + K_{SC}^{22} + M_t^{22} \tag{4.9}$$

$$F^1 = F_G^1 + F_{PG}^1 + MU_{(n)^-} \tag{4.10}$$

$$F^2 = F_G^2 + F_{PG}^2 \tag{4.11}$$

$$M_{ij} = \int_\Omega \begin{bmatrix} \varphi_i\varphi_j & 0 & 0 \\ 0 & \varphi_i\varphi_j & 0 \\ 0 & 0 & \varphi_i\varphi_j \end{bmatrix} d\Omega \qquad (jump\ term) \tag{4.12}$$

$$M_{t,ij}^{kl} = \int_{S_n} \begin{bmatrix} \varphi_i\varphi_j N_k \frac{\partial N_l}{\partial t} & 0 & 0 \\ 0 & \varphi_i\varphi_j N_k \frac{\partial N_l}{\partial t} & 0 \\ 0 & 0 & \varphi_i\varphi_j N_k \frac{\partial N_l}{\partial t} \end{bmatrix} d\Omega dt \tag{4.13}$$

Integrating the above expression in time,

$$M_{t,ij}^{11} = \frac{1}{2}\int_\Omega \begin{bmatrix} \varphi_i\varphi_j & 0 & 0 \\ 0 & \varphi_i\varphi_j & 0 \\ 0 & 0 & \varphi_i\varphi_j \end{bmatrix} d\Omega = \frac{1}{2}M \tag{4.14}$$

$$M_{t,ij}^{12} = \frac{1}{2}\int_\Omega \begin{bmatrix} \varphi_i\varphi_j & 0 & 0 \\ 0 & \varphi_i\varphi_j & 0 \\ 0 & 0 & \varphi_i\varphi_j \end{bmatrix} d\Omega = \frac{1}{2}M \tag{4.15}$$

$$M_{t,ij}^{21} = -\frac{1}{2}\int_\Omega \begin{bmatrix} \varphi_i\varphi_j & 0 & 0 \\ 0 & \varphi_i\varphi_j & 0 \\ 0 & 0 & \varphi_i\varphi_j \end{bmatrix} d\Omega = -\frac{1}{2}M \tag{4.16}$$

$$M_{t,ij}^{22} = \frac{1}{2}\int_\Omega \begin{bmatrix} \varphi_i\varphi_j & 0 & 0 \\ 0 & \varphi_i\varphi_j & 0 \\ 0 & 0 & \varphi_i\varphi_j \end{bmatrix} d\Omega = \frac{1}{2}M \tag{4.17}$$

Using these relations, the system of equations can be rewritten in the form

(4.18) $$\frac{1}{2}M\begin{bmatrix}-I & I\\ I & I\end{bmatrix}\begin{bmatrix}U_{(n)+}\\ U_{(n+1)-}\end{bmatrix}+\begin{bmatrix}K_G^{11} & K_G^{12}\\ K_G^{21} & K_G^{22}\end{bmatrix}\begin{bmatrix}U_{(n)+}\\ U_{(n+1)-}\end{bmatrix}+...+=\begin{bmatrix}F^1\\ F^2\end{bmatrix}$$

If can be seen in the above equation that the temporal coupling in the mass term $M$ is non symmetric. This leads to the explicit algorithm based on this process unconditionally unstable. However, this coupling can be transformed in to a symmetric coupling by using the following preconditioner matrix.

(4.19) $$2\begin{bmatrix}I & -I\\ I & I\end{bmatrix}$$

resulting

(4.20) $$\begin{bmatrix}M & 0\\ 0 & M\end{bmatrix}\begin{bmatrix}U_{(n)+}\\ U_{(n+1)-}\end{bmatrix}+.....+=2\begin{bmatrix}I & -I\\ I & I\end{bmatrix}\begin{bmatrix}F^1\\ F^2\end{bmatrix}$$

We incorporate the preconditioning described above using the following modified weighting functions

(4.21) $$\widehat{U}^h(x,y,t)=\sum_{j=1}^{n_{np}}\varphi_j(x,y)\Big(\Big(N_1(t)-N_2(t)\Big)\widehat{U}^h_{j;(n)+}+\widehat{U}^h_{j;(n+1)-}\Big)$$

## 5. NUMERICAL RESULTS

In this section we show some numerical results obtained with the presented methods. We will refer to SUPG as the generalized Streamline Upwind Petrov-Galerkin method for the case of semi-discrete formulation, presented in[8] and STPG as the correspondent space-time formulation; DCL and DCQ are the discontinuity-capturing methods, linear and quadratic, respectively; CAU-ST, G-ST, CAU-SD and G-SD are the CAU and Galerkin methods, with *ST* and *SD* standing for space-time and semi-discrete versions, respectively. In the semi-discrete formulation presented, the Crank-Nicolson scheme was used to approximate the temporal derivatives.

The first example is the well know *dam break* problem, which consists of a wall separating two undisturbed water levels that is suddenly removed

(Figure 5.1). Friction effects are neglected and the spatial discretization is given by a 100 linear one-dimensional elements mesh.

Figure 5.2 show the results for $t$=7.5, using a time step $\Delta t = 0.1$, and compares the solutions obtained with the G, STPG and CAU methods, in the space-time and semi-discrete formulation. As expected, the G, STPG and SUPG solutions presents some oscillations, wich disappear when the CAU operator is used. Figure 5.3 and 5.4 show the results for $t = 2.5$ and $t = 7.5$, wiht the time steps $\Delta t = 0.5$ and $\Delta t = 0.1$ using both, the CAU and the DCL operators.

Figure 5.5 compares the CAU, DCL and DCQ operators in the space-time formulation.

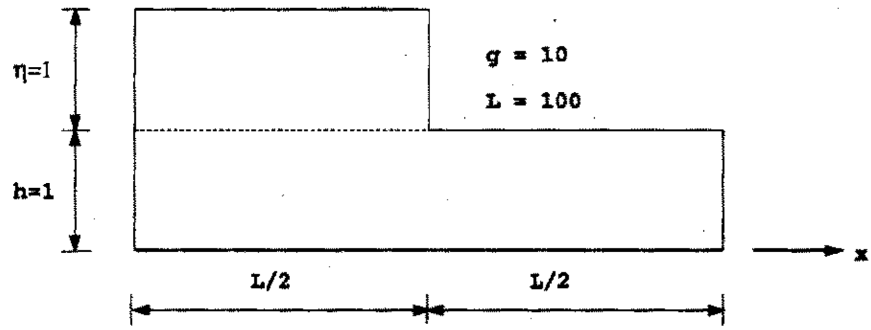


Figure 5.1. Dam break problem.

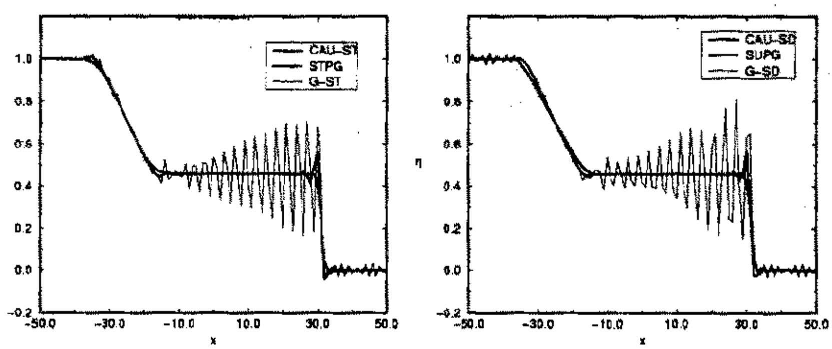


Figure 5.2. Solution for time $t = 7.5$, with $\Delta t = 0.1$.

The second example, illustrated by Figure 5.6, is the problem of a reflecting wave in a frictionless horizontal channel of length $L = 500$, discretized with 10 elements. The channel is open at the inflow boundary and closed at the opposed boundary. The system is subjected to a boundary condition at point A, raising the water level suddenly from the initial state of rest ($H = 10$) to $H = 10.1$, within one time step. The results can be seen in

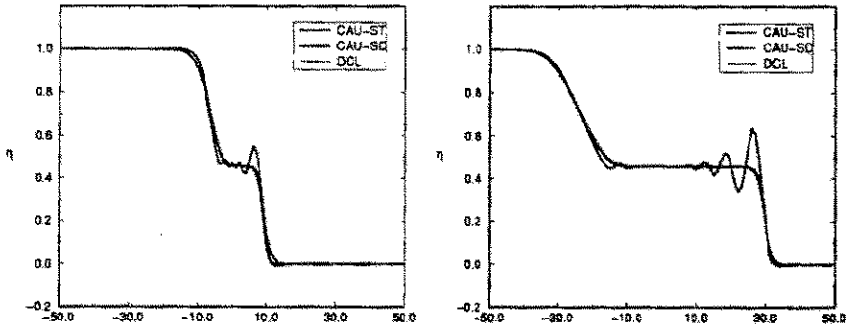


Figure 5.3. Solution for time $t = 2.5$ and $t' = 7.5$, with $\Delta t = 0.5$.

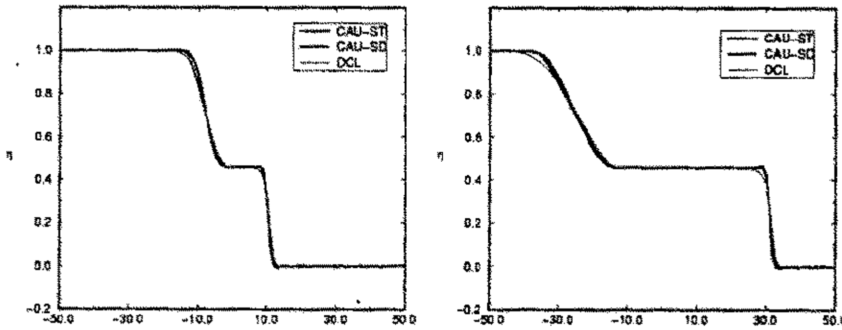


Figure 5.4. Solution for time $t = 2.5$ and $t' = 7.5$, with $\Delta t = 0.1$.

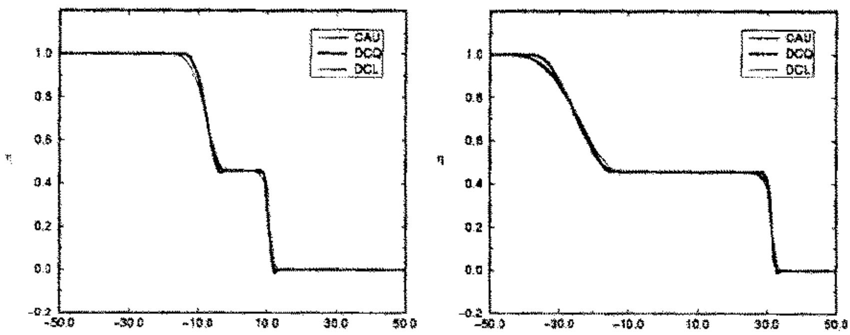


Figure 5.5. Solution for time $t = 2.5$ and $t' = 7.5$, with $\Delta t = 0.1$.

Figures 5.7 - 5.8. In these figures, the time-history responses for the water surface elevation at point $B$ are depicted.

Figure 5.7 show the curves with the space-time and semi-discrete formulations for the a time step $\Delta t = 10$. CAU-ST solution reaches the rectangular

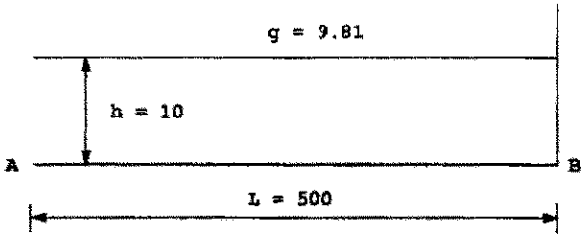


Figure 5.6. Reflecting wave in a frictionless channel.

form while the others solutions present some oscillations. Figure 5.8 presents results for the time steps $\Delta t = 10$ and $\Delta t = 1$ respectively, obtained with CAU-ST, CAU-SD and DCQ. The CAU-ST presents the best rectangular form.

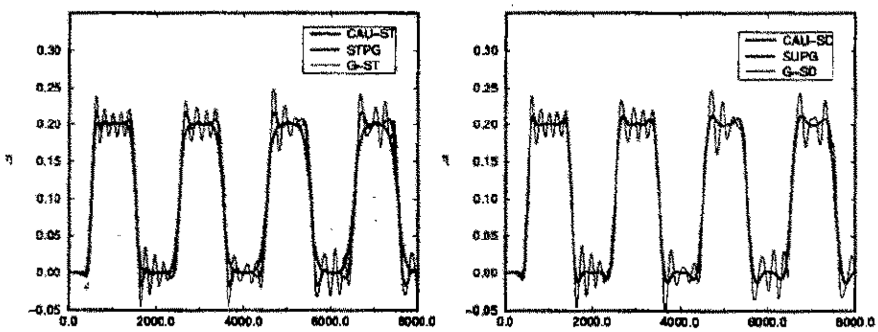


Figure 5.7. Solution at point B, with $\Delta t = 10$.

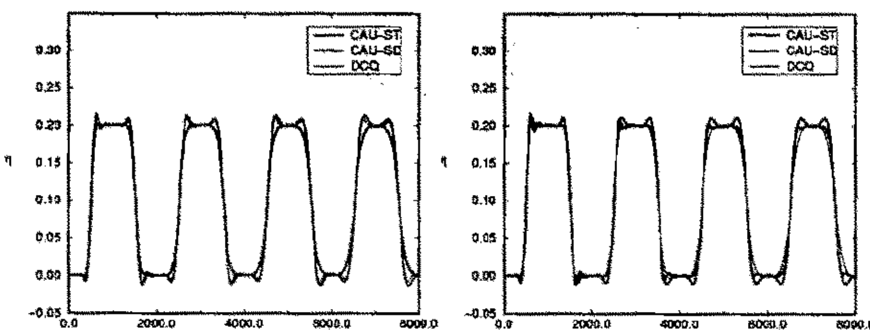


Figure 5.8. Solution at point B, with $\Delta t = 10$ and $\Delta t = 1$, respectively .

## 6. CONCLUSIONS

In this work a *STPG* model with shock-capturing operators was derived for problems governed by the shallow water equations. Piecewise linear approximations, continuous in space and discontinuous in time, were used. In addition, the correspondent semi-discrete versions were presented.

From the previously presented examples we can conclude that:

1) The space-time formulation allows the use of larger time-steps when compared with the correspondent semi-discrete formulation, but at the cost of duplicating the number of the equations. Nevertheless, using the preconditioning scheme previously presented this system of equations can be split in two coupled systems and solved iteratively.

2) For both examples the additional stability engendered by the shock-capturing terms provided by the CAU or DC methods eliminates the remaining oscillations still observed when the Galerkin or *SUPG* formulations were used.

3) Concerning accuracy and stability, the *CAU* method performs slightly better than the *DC* methods.

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