PESQUIMAT
Revista del Instituto de
Investigación de la Facultad
de Ciencias Matemáticas de la U.N.M.S.M
Vol. N°1.pp 01-26 Lima – Perú Julio-1998

HIDDEN REGULARITY FOR A STRONGLY NON LINEAR WAVE EQUATION

JAIME E. MUÑOZ RIVERA[1]

LNCC - Rio de Janeiro - Brazil

ABSTRACT

In this paper we consider the nonlinear wave equation,

$$\begin{cases} u'' - \Delta u + f(u) = V & \text{in } Q = \Omega \text{x}]0,T[; \\ u(0) = u_0, u'(0) = u_1 & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \sum = \Gamma \text{x}]0,T[\end{cases} \qquad (*)$$

where f is a continuous function satisfying

$$\limsup_{|S| \to +\infty} \frac{f(s)}{s} > -\infty \qquad (**)$$

and Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary Γ. We prove that there exist a solution for (*) that satisfies the regularity conditions: $\frac{\partial u}{\partial \eta} \in L^2(\sum)$.

---

[1]**Investigador del Laboratorio Nacional de Computación Científica (LNCC) de Rio de Janeiro - Brazil**

Moreover we have that there exist a constant $C > 0$ such that,

$$\left|\frac{\partial u}{\partial \eta}\right| \leq C\left\{E(0) + |V|_Q^2\right\} \qquad (***)$$

1. INTRODUCTION

Let Ω be an open bounded set of $\mathbb{R}^n$, with boundary Γ of class C^2. Set $Q = \Omega\text{x}]0,T[$ and $\sum = \Gamma\text{x}]0,T[$. We will denote by $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_Q$ the inner product of $L^2(\Omega)$ and $L^2(Q)$respectively and by $|.|_\Omega, |.|_Q$ and $\|.\|$, the norms in $L^2(\Omega)$,$L^2(Q)$, and $H_0^1(\Omega)$ respectively. We consider the nonliner problem:

$$\begin{cases} u'' - \Delta u + f(u) = V & \text{in } Q; \\ u(0) = u_0, u'(0) = u_1 & \text{in } \Omega; \\ u(x,t) = 0 & \text{on } \sum \end{cases} \qquad (1.1)$$

In J.L.Lions [1] was study the hidden regularity for system (1.1) when $f(s) = s^3$ and more generality for a $f(s) = s|s|^e$, where $e \geq 0$. In this work we are going to study the hidden regularity for the solution of the problem (1.1) when f is a continuous function satisfying,

$$\limsup_{|S|\to+\infty} \frac{f(s)}{s} > -\infty \tag{1.2}$$

That is, we will show that there exist a solution u of the above problem such that the normal derivate of u belongs to $L^2(\sum)$. Moreover we will prove that there exist a constant $C > 0$ such that:

$$\left|\frac{\partial u}{\partial \eta}\right|_{\sum} \leq CE_0 \tag{1.3}$$

where E_0 is the initial energy of the system (1.1).

$$\left(E_0 = \frac{1}{2}|u_1|_\Omega^2 + \int_\Omega G(u_0)\,dx\right)$$

where $G(s) = \int\limits_0^s f(\eta)\,d\eta$.

2. EXISTENCE AND HIDDEN REGULARITY

First of all we are going to construct a sequence of real numbers $(s_\nu)_{\nu\in N}$ and $(s_{-\nu})_{\nu\in N}$ satisfying the following conditions,

$$s_\nu \geq \nu, \forall \nu \in \mathbb{N}, |f(s_\nu)| \leq C + |f(s)|, \forall s \geq \nu \tag{2.1}$$

$$s_{-\nu} \geq -\nu, \forall \nu \in \mathbb{N}, |f(s_\nu)| \leq C + |f(s)|, \forall s \geq -\nu \tag{2.2}$$

This sequences are going to play and important role in the sequel.

LEMMA 2.1.- Let's f be a continuous function defined in $\mathbb{R}$, then there exists a sequence of real numbers, $(s_\nu)_{\nu\in N}$, and a positive constant C, independing of ν_1 satisfying condition (2.1) and (2.2).

PROOF.- Let's consider the following problem.

$$I_\nu = \inf \{|f(s)|; s \geq \nu\} \tag{2.3}$$

If for all $\nu \in \mathbb{N}$, there exist $s\nu \geq \nu$ such that $f(s_\nu) = I_\nu$, then this sequence satisfies condition (2.1). Now we can suppose that there exist a v_0 such that,

$$f(s) > I_{V_0} \quad \text{for all} \quad s \geq v_0$$

This relations emply that $I_\nu = I_{\nu_0}$ for all $\nu \geq \nu_0$. Let us put $I_0 = I\nu_0$.Since $I_0 = \inf |s(s)|\,; s \geq V_0$, there exists a sequence $(t_k)_{k\in\mathbb{N}}$ such that,

$$f(\bar{t}_k) \to I_0 \tag{2.4}$$

from the continuity of f we conclude that t_k is not bounded, then there exist a subsequence $(t_{k_\nu})_{\nu\in\mathbb{N}}$ satisfying:

$$t_{k_\nu} \geq V\,, \quad \nu \in \mathbb{N} \tag{2.5}$$

Let us put $s_\nu = t_{k_\nu}$, from (2.4) we obtain that there exist a constant C (independing of ν) such that $|f(s_\nu)| = |f(t_{k_\nu})| \leq C$, finally from (2.5) we conclude that $(s_\nu)_{\nu\in\mathbb{N}}$ satisfies condition (2.1). By the same arguments we can prove the existence of a sequence $(s_\nu)_{\nu\in\mathbb{N}}$ satisfying condition (2.2), only consider the problem

$$I_{-\nu} = \inf |f(s)|\,; s \leq -\nu,$$

and the result follows.

With the sequences $(s_\nu)_{\nu\in\mathbb{N}}$ and $(s_{-\nu})_{\nu\in\mathbb{N}}$ constructed in Lemma 2.1 we define a sequence $(f_\nu)_{\nu\in\mathbb{N}}$ of continuous function in the following way:

$$f_\nu(s) = \begin{cases} f(s) & \text{if} \quad s_\nu \leq s \leq s_\nu; \\ f(s_\nu) & \text{if} \quad s \geq s_\nu; \\ f(s_{-\nu}) & \text{if} \quad s_\nu \leq s_{-\nu}. \end{cases} \tag{2.6}$$

As a consequence of Lemma 2.1 we have that the sequence $(f_\nu)_{\nu\in\mathbb{N}}$ satisfies the following properties:

$$|f_\nu(s)| \leq C + |f(s)| \qquad \text{for all } \nu \tag{2.7}$$

$$f_\nu \to f \quad \text{uniformly on bounded sets} \tag{2.8}$$

LEMMA 2.2.- Let's f be a continuous function satisfying condition (1.2), and $(f_\nu)_{\nu\in\mathbb{N}}$, the sequence defined in (2.6). Then there exist a positive constant C_0 such that.

$$sf_\nu(s) \geq -C_0\left(s^2+1\right) \qquad \forall s \in \qquad \forall v \geq v_0 \tag{2.9}$$

$$\int_0^t f_\nu(s)\,ds \geq -C_0\left(t^2 + 2|t|\right) \tag{2.10}$$

$$\left|\int_0^t f_\nu(s)\,ds\right| \le \frac{1}{2}C_0\left|t(t+3)\right| + \int_0^t f(s)\,ds \tag{2.11}$$

PROOF.- First of we are going to prove that there exists a positive constant C_0, such that

$$f(s) \ge -C_0 s \quad \forall s \ge N \quad \text{and} \quad f(s) \le -C_0 s \quad \forall s \le N \tag{2.12}$$

In fact, if $\liminf s^{-1} f(s) = +\infty$ the expression (2.12) is valid. Now we can suppose that $\liminf s^{-1} f(s) = x < +\infty$, then for $\mathcal{E} > 0$,there exist $N > 0$ such that $s^{-1} f(s) > x^{-\mathcal{E}}$, for $|s| \ge N$. Let us take $C = \sup\{|f(s)|; |s| \le N\}$, $C_2 = \sup\{|sf(s)|; |s| \le N\}$, and put $C_0 = \max\{C, C_1, C_2, |x - \mathcal{E}|\}$ where C is the constant in (2.7), certainly for this C_0, condition (2.12) is valid. Finally multiplying the relations in (2.12) by $s\,(|s| \ge N)$, we have by the definition of C_0, that the first part of (2.9) is valid. The second part of (2.9) follows from (2.1), (2.2), (2.6) and also, the definition of C_0, for $v_0 = N$.

In order to prove (2.11), let us note that from (2.12) follows that:

$$f_\nu(s) \ge -C_0(s+1) \quad \forall s \ge 0 \quad \text{and} \quad f_\nu(s) \le -C_0(s-1) \quad \forall s \le 0 \tag{2.13}$$

Integrating this expression we obtain (2.10). In order to obtain (2.11) let us consider relation (2.7) then we have:

$$\left|\int_0^t f_\nu(s)\,ds\right| \le \int_0^{t^0} |f_\nu(s)|\,dsC\,|t| + \int_0^t |f(s)\,ds| \quad \forall t \in \mathbb{R} \tag{2.14}$$

From (2.12) we obtain that $f(s) \ge -C_0(s+1) \quad \forall s \ge 0$, which imply that $|f(s)| \le f(s) + 2_{\dot{c}}0(s+1) \quad \forall s \ge 0$. Then we have:

$$\int_0^t |f(s)\,ds| \le \int_0^t f(s)\,ds + C_0 t\,(t+2) \quad \forall t{\ge}0 \tag{2.15}$$

and since $f(s) \le |f(s)|$ we obtain,

$$\int_0^t |f(s)|\,ds \le \int_0^t f(s)\,ds \quad \forall t{\ge}0 \tag{2.16}$$

Finally from (2.15) and (2.16) we obtain (2.11).

Let us denote by $G_\nu(t) = \int_0^t f_\nu(s)\,ds$, then we have that

$$G_\nu \to G \quad \text{uniformly on bounded sets.} \tag{2.17}$$

Before to prove the main result of this paper we will prove an identity that will be fundamental in that follows.

LEMMA 2.3.- Le h be a continuous function.Let $q = (q_k)$ a field of vectors of class $\left[C^1\left(\overline{\Omega}\right)\right]^n$. Then for all W satisfying.

$$W \in L^2\left(0,T;H_0^1(\Omega)\,\eta H^2(\Omega)\right), h(W) \in L^1(Q) \tag{2.18}$$

$$W' \in L^2\left(0,T;H_0^1(\Omega)\right) \tag{2.19}$$

$$W'' \in L^2\left(0,T;L^2(\Omega)\right) \tag{2.20}$$

$$\begin{cases} W'' - \Delta W + h(W) = V & \text{in} \quad Q, \\ W(0) = W_0, W'(0) = W_1 & \text{in} \quad \Omega; \\ W(x,t) = 0 & \text{on} \quad \sum. \end{cases} \tag{2.21}$$

Where H is the primitive of h. Then we have,

$$\frac{1}{2}\int_{\Sigma} q_k n_k \left|\frac{\partial W}{\partial \eta}\right|^2 d\Sigma = \left[\left(W'(t), q_k \frac{\partial W(t)}{\partial x_k}\right]_0^T + \right.$$
$$+ \int_Q \frac{\partial q_k}{\partial x_j}\left\{\left|W'\right|^2 - |\nabla W|^2 - 2H(W)\right\} dx\, dt$$
$$+ \int_Q \frac{\partial q_k}{\partial x_j} \times \frac{\partial W}{\partial x_k} \times \frac{\partial W}{\partial x_j} dx\, dt - \int_Q V_{q_k} \frac{\partial W}{\partial x_k}$$

PROOF.-Let us multiply (2.21), by $q_k \frac{\partial W}{\partial x_k}$, then we have that:

$$\int_Q \{W'' - \Delta W + h(W)\}\, q_k \frac{\partial W}{\partial x_k} dx\, dt = \int_Q V q_k \frac{\partial W}{\partial x_k} dx\, dt \tag{2.22}$$

Let us denote by:

$$I_1 = \int_Q W'' q_k \frac{\partial W}{\partial x_k} dx\, dt$$
$$I_2 = \int_Q \Delta W q_k \frac{\partial W}{\partial x_k}$$

then we have:

$$I_1 = \left[\left(W'(t), q_k \frac{\partial W}{\partial x_k}(t)\right)\right]_0^T - \int_0 W' q_k \frac{\partial W'}{\partial x_k} dx\, dt$$
$$= \left[\left(W'(t), q_k \frac{\partial W(t)}{\partial x_k}\right)\right]_0^T - \frac{1}{2}\int_0 q_k \frac{\partial \left|W'\right|^2}{\partial x_k} dx\, dt$$

from where we obtain that:

$$I_1 = \left[\left(W'(t), q_k \frac{\partial W(t)}{\partial x_k}\right)_\Omega\right]_0^T + \frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k} |W'|^2 \, dx \, dt \qquad (2.23)$$

on the other hand we have that:

$$I_2 = -\int_Q \frac{\partial W}{\partial x_j} \times \frac{\partial}{\partial x_j}\left\{q_k \frac{\partial W}{\partial x_k}\right\} dx \, dt + \int_\Sigma q_k \frac{\partial W}{\partial x_k} \times \frac{\partial W}{\partial \eta} d\Sigma$$

$$= -\int_Q \frac{\partial W}{\partial x_k} \frac{\partial q_k}{\partial x_k} \times \frac{\partial W}{\partial x_k} dx \, dt - \int_Q \frac{\partial W}{\partial x_j} \times \frac{\partial^2 W}{\partial x_k \partial x_j} \times q_k \, dx \, dt.$$

$$+ \int_\Sigma q_k \frac{\partial W}{\partial x_k} \times \frac{\partial W}{\partial \eta} d\Sigma.$$

$$= -\int_Q \frac{\partial W}{\partial x_k} \times \frac{\partial q_k}{\partial W_j} \times \frac{\partial W}{\partial x_k} dx \, dt - \frac{1}{2}\int_Q qk \frac{\partial}{\partial x_k} |\nabla W|^2 \, dx \, dt + \int_\Sigma q_k \frac{\partial W}{\partial x_k} \times \frac{\partial W}{\partial \eta} d\Sigma.$$

But since $W = 0$ on $\sum$ we have that:

$$\frac{\partial W}{\partial x_k} = \eta_k \frac{\partial W}{\partial \eta} \text{ on } \sum.$$

and

$$|\nabla W|^2 = \left|\frac{\partial W}{\partial \eta}\right|^2 \text{ on } \sum.$$

we have:

$$
\begin{aligned}
I_2 = & -\int_Q \frac{\partial W}{\partial x_j} \times \frac{\partial W}{\partial x_k} \times \frac{\partial q_k}{\partial x_j} dx\, dt + \frac{1}{2} \int_Q \frac{\partial q_k}{\partial x_k} |\nabla W|^2 dx\, dt. \\
& + \frac{1}{2} \int_{\Sigma} q_k \eta_k \left| \frac{\partial q_k}{\partial x_k} \right|^2 d\Sigma
\end{aligned}
\tag{2.24}
$$

finally since

$$
\int_Q h(W)\, q_k \frac{\partial W}{\partial xk} dx\, dt = \int_Q \frac{\partial}{\partial x_k} H(W)\, q_k = \int_Q H(W) \frac{\partial}{\partial x_k} q_k. \tag{2.25}
$$

we have from (2.22), (2.23), (2.24) and (2.25) that the result follows.

REMARK 2.4.- From Lemma 2.3 taking $q = (q_k)$ a field of vectors of class $\left[C^1\left(\overline{\Omega}\right)\right]^n$, such that

$$
q_k = \eta_k \text{ on } \textstyle\sum,
$$

and putting:

$$
C = \sup\left\{ |q_k(x)|, \left|\frac{\partial q_k}{\partial x_k}(x)\right|; k, j = 1, ..., n \text{ and } \mathbf{x} \in \overline{\Omega} \right\}
$$

we have:

$$\frac{1}{2}\int_{\Sigma}\left|\frac{\partial W}{\partial \eta}\right|^{2} d\Sigma \leq 2n\sup_{[0,T]} J(t) +$$
$$+CnT\sup_{[0,T]} J(t) + Cn\int_{Q}|H(W)|\,dx\,dt +$$
$$+2C(\sup J(t)) + \frac{Cn}{2}|V|_{Q}^{2} + CT\sup J(t)$$

From where we have:

$$\begin{aligned}\frac{1}{2}\int_{\Sigma}\left|\frac{\partial W}{\partial \eta}\right| d\Sigma \leq C(n+1)(2+T)\sup_{[0,T]} J(t) +\\ +Cn\frac{1}{2}|V|_{Q}^{2} + \int_{Q}|H(W)|\,dx\,dt\end{aligned} \tag{2.26}$$

REMARK 2.5.- From (2.21) we have that

$$W'' - \Delta W + h(W) + bW = V + bW$$

mutiplying this expression by W' and integrating in Ω we have:

$$\frac{d}{dt}\left\{J(t) + \int_{Q} H(W) + b|W|^{2}\,dx\right\} = (V, W')_{\Omega} + b(V, W')_{\Omega}$$

where $J(t) = \frac{1}{2}\left\{|W'(t)|^{2} + \|W(t)\|^{2}\right\}$. If we put $C_0 = \max\{1, b+1, bc^{3}\}$

(where c is such that $|\cdot|_\Omega \leq \|\cdot\|$)we obtain, after integrate from 0 to t, that:

$$J(t) + \int_Q \left\{H(W) + b(W)^2\right\} dx \leq \frac{1}{2}|V|_Q^2 + 2C_0\mathbf{E}(0) + C_0\int_0^1 J(s)\,ds \qquad (2.27)$$

where $\mathbf{E}(t)$ is the energy associate with system (2.21), that is:

$$\mathbf{E}(t) = J(t) + \int_\Omega H(W(x,t))\,dx$$

REMARK 2.6.- Multiplying $(2.21)_1$ by W, integrating in Q, and applying Green's formuls, we have that:

$$\int_Q Wh(W)\,dx\,dt = \int_Q WV\,dx\,dt + \int_0^T |W'(t)|_\Omega^2\,dt - \\ - \int_0^T \|W(t)\|^2\,dt - (W'(t), W(t))_0^T$$

from where we have that:

$$\int_Q Wh(W)\,dx\,dt \leq \frac{1}{2}|V|_Q^2 + (3T + 2C)\sup_{[0,T]} J(t) \qquad (2.28)$$

(where C is such that $|\cdot|_\Omega \leq C\|\cdot\|$).

Now we are condition to prove the main result of this paper:

THEOREM 2.7.- Let (u_0, u_1, V) be an elemente of the space

$H_0^1(\Omega)\times L^2(\Omega)\times L^2(Q)$, and let f b a continuous function such that $G(u_0)\in L^1(Q)$. Then there exist a function $u : Q\to\mathbb{R}$ satisfying.

$$u\in L^\infty\left(0,T : H_0^1(\Omega)\right), u'\in L^\infty\left(0,T; H^2(\Omega)\right) \tag{2.29}$$

$$\begin{cases} u''-\Delta u+f(u)=V & \text{in } Q; \\ u(0)=u_0, u'(0)=u_1 & \text{in } \Omega; \\ u(x,t)=0 & \text{on } \Sigma \end{cases} \tag{2.30}$$

REMARK 2.8.- We are proving here that, exist one solution satisfying the last two conditions. We don't know if all solution of (1.1) satisfies this regularity result. This is an open question.

PROOF OF THEOREM 2.7.- Let $(\rho_\mu)_{\mu\in\mathbb{N}}$ be a regularizant sequence on $\mathbb{R}$. That is: $\rho_\mu\in C^\infty(\mathbb{R}.), \forall\mu\in\mathbb{N}$ and:

$$\rho_\mu(s)\geq 0 \quad \forall s\in\mathbb{R} \quad \text{and} \quad \text{sopp }(\rho_\mu) \subset \left]-\frac{1}{\mu},\frac{1}{\mu}\right[\tag{2.34}$$

$$\int_{\mathbb{R}}\rho_\mu(s)\,ds=1 \quad \forall\mu\in\mathbb{N} \tag{2.35}$$

Let us denote by $(f_{\nu\mu})_{\mu\in\mathbb{N}}$ the sequence of bounded function defined by:

$$f_{\nu\mu} = f_\nu * \rho_\mu \text{ for a fixed } \nu.$$

Then we have f_{nm} is a C^∞ bounded function we now consider the following aproximated problem.

$$\begin{cases} u''_{\nu\mu} - \Delta u_{\nu\mu} + f(u_{\nu\mu}) = V & \text{in} \quad Q; \\ u_{\nu\mu}(0) = u_0, u'_{V_\mu} = u_1 & \text{in} \quad \Omega; \\ u_{\nu\mu}(x,t) = 0 & \text{on} \quad \Sigma \end{cases} \tag{2.36}$$

As well known that for every $(u_0, u_1, V) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(Q)$ there exist an unique solution for system (2.33). In order to obtain the existence of a solution for the system (1.3), let us suppose that, V, u_0, u_1 be test function, then we have that:

$$u_{\nu\mu} \in L^\infty\left(0, T; H_0^1(\Omega) \eta H^2(\Omega)\right). \tag{2.37}$$

$$u_{\nu\mu} \in L^\infty\left(0, T; H_0^1(\Omega)\right) \tag{2.38}$$

$$u_{\nu\mu} \in L^{\infty}\left(0,T;H^{2}(\Omega)\right) \tag{2.39}$$

From Remark (2.4) we have that the normal derivate of $u_{\nu\mu}$, satisfies the following inequality:

$$\frac{1}{2}\int_{\Sigma}\left|\frac{\partial u_{\nu\mu}}{\partial\eta}\right|^{2}d\Sigma \leq C(n+1)(2+T)\sup J_{\nu\mu}(t)+ \\ +cn\left\{\frac{1}{2}|V|_{Q}^{2}+\int_{Q}|G_{\nu\mu}(u_{\nu\mu})|\right\}dx\,dt. \tag{2.40}$$

Where by $J_{\nu\mu}(t)$ we are denoting the quadratic term associated to system (2.36) that is:

$$J_{\nu\mu}(t)=\frac{1}{2}|u_{\nu\mu}(t)|_{\Omega}^{2}+\frac{1}{2}\|u_{\nu\mu}(t)\|^{2}$$

By Remarks 2.5, we have that,

$$J_{\nu\mu}(t)+\int_{\Omega}G_{\nu\mu}(u_{V\mu})+b|u_{\nu\mu}|^{2}dx \leq \frac{1}{2}|V|_{Q}^{2}+ \\ +2C_{Q}\ E_{\nu\mu}(0)+C_{0}\int_{0}^{T}J_{\nu\mu}(s)\,ds. \tag{2.41}$$

and since b is a positive number, and $G_{\nu\mu}$ is uniformly bounded for all $\mu \in \mathbb{N}$, and a fixed ν, we have by Gronwall inequality that there exist a constand C_ν such as:

$$J_{\nu\mu}(t) = \frac{1}{2}\left|u'_{\nu\mu}(t)\right|^2_\Omega + \frac{1}{2}\left\|u_{\nu\mu}(t)\right\|^2 \leq C_\nu. \tag{2.42}$$

where $E_{\nu\mu}(t)$ is the energy associated to system (2.36) with non quadratic term $G_{\nu\mu}(u_{\nu\mu})$.

Finally from Remark 2.6 we obtain that:

$$\begin{aligned} &\int_Q u_{\nu\mu} f_{V_\mu}(u_{\nu\mu})\, dx \leq + \\ &+\tfrac{1}{2}|V|^2_Q + (3T + 2C)\sup_{[0,T]} J_{V_\mu}(t). \end{aligned} \tag{2.43}$$

Relation (2.41), (2.42), and (2.43) are valid when V, u_0 and u_1 are test function. If we take a sequence of test function (V_m, u_{0m}, u_{1m}) satisfying,

$$(V_m, u_{0m}, u_{1m}) \rightarrow (V, u_0, u_1) \text{ strongly} \quad \text{in } L^2(Q) \times H^1_0(\Omega) \times L^2(\Omega)$$

certainly we have that the corresponding solutions $u_{\nu um}$ converge to $u_{\nu u}$ solution of system (2.36) , when the datas V, u_0 and u_1 are $L^2(Q)$, $H^1_0(\Omega)$ and $L^2(\Omega)$

respectively. Moreover we have:

$$u_{\nu u m} \rightarrow u_{\nu u} \text{ strongly} \quad \text{in } L^{\infty}\left(O,T;H_0^1(\Omega)\right). \tag{2.44}$$

$$u_{\nu u m} \rightarrow u_{\nu u} \text{ strongly} \quad \text{in } L^{\infty}\left(O,T;L^2(\Omega)\right) \tag{2.45}$$

From (2.44) and (2.45) we conclude that relations (2.40), (2.41), (2.42) and (2.43) are valid when (V,u_0,u_1) belongs to $L^2(Q)\times H_0^1(\Omega)\times L^2(\Omega)$ and $u_{\nu u}$ is solution of (3.36).

On the other hand, by (2.42) we obtain that there exists a subsequence of $(u_{\nu\mu})_{\mu\in\mathbb{N}}$ satisfying:

$$u'_{\nu\mu} \rightarrow u_{\nu} \text{weak star in } L^{\infty}\left(O,T;H_0^1(\Omega)\right) \tag{2.46}$$

$$u'_{\nu\mu} \rightarrow u'_{\nu} \text{ weak star in } L^{\infty}\left(O,T;L^2(\Omega)\right) \tag{2.47}$$

from (2.46) and (2.47) we have that there exist another subsequence (that we still denoting in the same way) such that,

$$u_{\nu\mu} \rightarrow u_{\nu} \text{ strongly} \quad \text{in } L^2(Q). \tag{2.48}$$

$$u_{\nu\mu} \to u_\nu \text{ a. e. in } Q. \tag{2.49}$$

$$f_{\nu\mu}(u_{\nu\mu}) \to f_\nu(u_\nu) \text{ a. e. in } Q. \tag{2.50}$$

$$G_{\nu\mu}(u_{\nu\mu}) \to G_\nu(u_\nu) \text{ a. e. in } Q. \tag{2.51}$$

Since $f_{\nu\mu}$ is bounded for all $\mu \in \mathbb{N}$ (ν fixed), then $G_{\nu\mu}$ is a Lipschitz's in $\mathbf{R}$, then by Lebesgue dominated convergence theorem we conclude that:

$$f_{\nu\mu}(u_{\mu\mu}) \to f_\nu(u_\nu) \text{ strongly in } L^2(Q). \tag{2.52}$$

$$G_{\nu\mu}(u_{\nu\mu}) \to G_\nu(u_\nu) \text{ strongly in } L^2(Q). \tag{2.53}$$

Now, from (2.48) and (2.52) we obtain:

$$u_{\nu\mu} \to u_\nu \text{ strongly in } L^\infty\left(O,T;H_0^1(\Omega)\right). \tag{2.54}$$

$$u'_{\nu\mu} \to u'_\nu \text{ strongly in } L^\infty\left(O,T;L^2(\Omega)\right). \tag{2.55}$$

Then by (2.40), (2.42) and (2.53) we obtain that there exist a subsequence of $\frac{\partial u}{\partial \eta}\nu\mu$, which we still denote in the same way and a element $\mathcal{X}$ in $L^2(\Sigma)$ such that:

$$\frac{\partial u_{\nu\mu}}{\partial \eta} \to \mathcal{X}\nu \text{ weak in } L^2\left(\sum\right) \tag{2.56}$$

But since:

$$\frac{\partial u_{\nu\mu}}{\partial \eta} \to \frac{\partial u_\nu}{\partial \eta} \text{ weak in } H^{-1}\left(O,T;H^{1/2}(\Gamma)\right).$$

we conclude that $\mathcal{X}_\nu = \frac{\partial u_\nu}{\partial \eta}$. From (2.54) and (2.55) we have in particular that:

$$J_{\nu\mu}(t) \to J_\nu(t) \text{ uniformly on } [O,T]. \tag{2.57}$$

$$E_{\nu\mu}(t) \to E_\nu(t) \text{ uniformly on } [O,T]. \tag{2.58}$$

Then form $(2.40), (2.53)$ and (2.57) we obtain that:

$$\begin{gathered}\tfrac{1}{2}\int_{\sum}\left|\tfrac{\partial u_\nu}{\partial \eta}\right|^2 d\sum \le C(n+1)(2+T)\sup_{[O,T]} J_\nu(t) + \\ +Cn\left\{\tfrac{1}{2}|V|_Q^2 + \int_Q |G_\nu(u_\nu)|\right\} dx\, dt.\end{gathered} \tag{2.59}$$

Now by $(2.41), (2.43), (2.48), (2.53), (2.57)$ and (2.58) we obtain:

$$\begin{gathered}J_\nu(t) + \int_\Omega \left\{G_\nu(u_\nu) + b|u_\nu|^2\right\} \le \\ \le \tfrac{1}{2}|V|_Q^2 + 2C_0E_\nu(0) + C_0\int_0^t J_\nu(s)\, ds.\end{gathered} \tag{2.60}$$

$$\int u_\nu f_\nu (u_\nu)\, dx\, dt \leq \frac{1}{2} |V|_Q^2 + (3T + 2C) \sup_{[O,T]} J_\nu (t). \tag{2.61}$$

From (2.10) and (2.59) taking b such that $G_\nu (u_\nu) + b |u_\nu|^2$ be positive, we obtain:

$$J_\nu (t) \leq \frac{1}{2} |V|_Q^2 + 2C_0 E_\nu (0) + C_0 \int_0^t J_\nu (s)\, ds. \tag{2.62}$$

Now by Gronwal's inequality we obtain that:

$$J_\nu (t) \leq \left\{ \frac{1}{2} |V|_Q^2 + 2C_0 E_\nu (0) \right\} e^{C_0 T}, \forall t \in [O, T]. \tag{2.63}$$

Since,

$$E_\nu (0) = \frac{1}{2} \left\{ \|u_0\|^2 + |u_1|_\Omega^2 \right\} + \int_\Sigma G_\nu (u_0)\, dx,$$

we conclude from (2.12) and the hypothesis of Theorem 2.6, that the second member of (2.63) is bounded by a constand $C_1 > 0$, independing of ν, then from (2.63) we have:

$$\sup_{[O,T]} J_\nu (t) \leq \left\{ \frac{1}{2} |V|_Q^2 + 2C_0 E_\nu (0) \right\} e^{C_0 T} \leq C_1. \tag{2.64}$$

Then we have that there exist a subsequence of $(u_\eta)_{\eta \in \mathbb{N}}$, that we still denote on the same way, and a element $u \in L^{\infty}(O,T;H_0^1(\Omega))$ such that $u' \in L^{\infty}(O,T;L^2(\Omega))$, satisfying:

$$u \in L^{\infty}\left(O,T;H_0^1(\Omega)\right) \text{ such that } u' \in L^{\infty}\left(O,T;L^2(\Omega)\right), \tag{2.65}$$

$$u_{\nu}' \to u' \text{ weak star in } L^{\infty}\left(O,T;L^2(\Omega)\right). \tag{2.66}$$

By (2.61) and (2.63) we obtain that:

$$\int_Q u_{\nu} f_{\nu}(u_{\nu})\, dx \leq 3C_2 e^{C_0 T}\left\{|V|_Q^2 + E_{\nu}(0)\right\} \leq C_3. \tag{2.67}$$

where $C_2 = \max\{n, T, C, C_0\}$. But from (2.9) we obtain that $|u_{\nu} f_{\nu}(u_{\nu})| \leq u_{\nu} f_{\nu}(u_{\nu}) + 2C_0(u_{\nu}^2 + 1)$, from where we have:

$$\int_Q |u_{\nu} f_{\nu}(u_{\nu})|\, dx \leq C_3 + 4C_0 \text{x} C_1 \text{ med } (Q) = C_4. \tag{2.68}$$

Then by (2.68) and from Theorem 1.1 of W.A. Strauss [4] we have that:

$$f_{\nu}(u_{\nu}) \to f_{(u)} \text{ strongly in } L^1(Q). \tag{2.69}$$

Then we conclude that u is a solution of problem (1.1). Finally from (2.59) and (2.63) we have that there exist a constant C_5 (independing of ν) such that:

$$\int_{\Sigma}\left|\frac{\partial u_\nu}{\partial \eta}\right| d\sum \leq C_s\left\{|V|_Q^2+E_\nu(0)+\int_Q |G_\nu(u_\nu)|\,dx\,dt\right\} \tag{2.70}$$

But from (2.60) and (2.64) we have that exist a constant C_6 such that,

$$\int_Q G_\nu(u_\nu)\,dx \leq C_8\left\{|V|_Q^2+E_\nu(0)\right\}. \tag{2.71}$$

Now from (2.10) we have that $G_\nu(t) \leq -C_0(t^2+2|t|)$, from where we conclude that $|G_\nu(t) \leq G_\nu(t)|+2C_0(t^2+2|t|)$ taking $t=u_n$ we have after to integrate in Q, that:

$$\int_Q G_\nu(u_\nu)\,dx\,dt \leq \int_Q G_\nu(u_\nu)\,dx\,dt+2C_0\int_Q u_\nu^2+2|u_\nu|\,dx\,dt. \tag{2.72}$$

On the other hand, there exist a constant C_7 such that:

$$\int_Q |u_\nu|+2|u_\nu|\,dx\,dt \leq C_7\sup_{[0,T]} J_\nu(t). \tag{2.73}$$

From (2.70), (2.71), (2.72) and (2.73) we obtain another constant, say C_8 such that:

$$\int_{\Sigma}\left|\frac{\partial u_\nu}{\partial \eta}\right| d\Sigma \leq C_8\left\{|V|_Q^2 + E_\nu(0)\right\} \tag{2.74}$$

Since the second member of (2.74) is bounded we obtain a subsecuence of,

$$\left(\frac{\partial u_\nu}{\partial \eta}\right)_{\nu\in\mathbb{N}}$$

and a element $\mathcal{X}$ in $L^2(\Sigma)$ such that:

$$\frac{\partial u_\nu}{\partial \eta} \rightarrow \mathcal{X} \ \text{ weak } \ \text{in } L^2(\Sigma). \tag{2.75}$$

But,

$$\frac{\partial u_\nu}{\partial \eta} \rightarrow \frac{\partial u_\nu}{\partial \eta} \text{ in } H^{-1}\left(0,T; W^{-1/p-2,p'}(\Gamma)\right),$$

Where $p > \frac{n}{2}$, then we have that $\mathcal{X} = \frac{\partial u}{\partial \eta}$, and letting $\nu \rightarrow \infty$ in (2.74) we have,

$$\int_{\Sigma}\left|\tfrac{\partial u}{\partial \eta}\right|^2 d\Sigma \leq C_8\left\{|V|_Q^2 + E(0)\right\} \qquad \text{Q.E.D.}$$

REFERENCES

1.- J. L. Lions - Hidden regularity in some nonlinear hyperbolic equations. *Matemática Aplicada e Computacional*, v.6, n.1, pp. 7-15, (1987).

2.- J. L. Lions - Quelques méthodes de résolution des problemes aux limites nonlineares, Paris. Dunod, (1969).

3.- J. L. Lions - Controlabilité exate de systemes distribues, vol. 1, College de France, Departament de Mathématiques, n. 3,, Rue d'Ulm 75005. Paris. France.

4.- W. A. Strauss - On weak solution of semilinear hyperbolic equations, *An, Acad. Brasil. Ciencias*, vol. 42, n,4, pp. 645 - 651, (1970).