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EXISTENCIA DE SOLUCIONES A PROBLEMAS ELÍPTICOS NO LOCALES CON DEPENDENCIA DEL GRADIENTE VÍA LA TÉCNICA DE PASO DE MONTAÑA

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Resumen: El objetivo principal de este trabajo es estudiar la solubilidad del problema elíptico no local

$$-M(\int_{\Omega}|\nabla u|^2)\Delta u = f(x,u,\nabla u)$$

con condición de frontera de Dirichlet cero en un dominio suave y acotado de $\mathbb{R}^n$, con $f:\Omega\to\mathbb{R}$ y $M:\mathbb{R}\to\mathbb{R}$ como funciones dadas.
Palabras clave: Problemas elípticos no locales, paso de montaña, métodos de iteración.

EXISTENCE OF SOLUTIONS TO NONLOCAL ELIPTIC PROBLEM WITH DEPENDENCE ON THE GRADIENT VIA MOUNTAIN–PASS TECHNIQUES

Abstract: The main goal of this work is to study the solvability of the nonlocal elliptic problem

$$-M(\int_{\Omega}|\nabla u|^2)\Delta u = f(x,u,\nabla u)$$

with zero Dirichlet boundary conditions on a bounded and smooth domain of $\mathbb{R}^n$, with $f:\Omega\to\mathbb{R}$ and $M:\mathbb{R}\to\mathbb{R}$ are given functions.
Key words: Nonlocal elliptic problems, mountain pass, iteration methods.

1. Introduction

The purpose of this article is to investigate the existence of solutions for the nonlocal elliptic problem

$$\begin{aligned} -M(\int_{\Omega}|\nabla u|^2)\Delta u &= f(x,u,\nabla u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$, $N\geq 3$ is a bounded smooth domain, $f:\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $M:\mathbb{R}\to\mathbb{R}$ are given functions. The equation (1.1) is not variational and when $M(t)=1$ was studied by several authors (See :???) using topological degree, methods of sub and supersolutions, etc. So the well developed critical point theory is of no avail for a direct attack to problem (1.1). In the present

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work we adapt the technique explored by De Figueiredo et al. [5]: we associate with the problem (1.1) a family of semilinear elliptic problems with no dependence on the gradient of the solution; this new problems are variational and we can apply the mountain–pass techniques, then we use an iterative scheme.
As it s well known, problem (1.1) is the stationary counterpart of the hyperbolic Kirchhoff equation

$$\begin{aligned} \rho u_{tt} - \left[\frac{P_0}{} + \frac{E}{2L}\int_0^L u_x^2 dx\right] u_{xx} = 0 \quad \text{in } (0,L)\times(0,\infty), \\ u(0,t) = 0 = u(L,t) \quad \text{on } (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } (0,L). \end{aligned} \tag{1.2}$$

that appeared at the first time in the work ok Kirchhoff [??], in 1883. The equation in (1.2) is called Kirchhoff equation and it extends the classical D'alembert wave equation, by considering the effects of the changes in the length of the strings during the vibrations.

The interest of the mathematicians on the so-called nonlocal problems like (1.1), (1.2) (nonlocal because of the presence of the term $M\big(\int_\Omega |\nabla u|^2 dx\big)$) has increased because they represent a variety of relevant physical situations and requires a nontrivial apparatus to solve them.
The paper is organized as follows: In section 2, we will give the existence of solutions for the system

$$\begin{aligned} -M(\int_\Omega |\nabla w|^2)\Delta u = f(x,u,\nabla w) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

for each $w \in H_0^1(\Omega)$. In section 3 we will study the solution for (1.1) using a iterative scheme and results of section 2.

2. Notations and Preliminaries

We will denote by C the general positive constant (the exact value may change from line to line). For convenience, we give the following hypotheses

(H.1) (i) A typical assumption for $M \in C^1(0,+\infty)$ is that there exists $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \in [0,+\infty[$

(ii) There exists $m_1 > m_0$ such that $M(t) = m_1$ $\forall t \geq t_0$ for some $t_0 > 0$

(H.2) We suppose that $f : \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a locally Lipschtiz continuous

(i) $\text{lím}_{t\to 0} \frac{f(x,t,\xi)}{t} = 0$ uniformly for all $x \in \overline{\Omega}, \xi \in \mathbb{R}^N$

(ii) There exist constants $a_1 > 0$ and $p \in \left(1, \frac{N+2}{N-2}\right)$ such that

$$|f(x,t,\xi)| \leq a_1(1+|t|^p) \quad \forall\xi \in \mathbb{R}^N, \quad t \in \mathbb{R}, \quad \xi \in \mathbb{R}^N$$

(iii) There exists constant $\theta > \text{máx}\left\{2, \frac{2}{m_0}\right\}$ and $T > 0$ such that

$$0 < \theta \quad (x,t,\xi) \leq tf(x,t,\xi) \quad \forall\xi \in \mathbb{R}^N, \quad |t| \geq T \in \mathbb{R}, \quad \xi \in \mathbb{R}^N$$

where

$$(x,t,\xi) = \int_0^t f(x,s,\xi)ds$$

(iv) There exist constant $a_2, a_3 > 0$ such that

$$(x, t, \xi) \geq a_2|t|^\theta - a_3 \quad \text{for all } x \in \overline{\Omega}, \quad t \in \mathbb{R}, \quad \xi \in \mathbb{R}^N$$

Observation 2.1. *From (i) and (ii) it follows that* $\theta \leq p+1$

(H.3) The function f satisfies

$$\begin{array}{ll}(i) & |f(x,t',\xi) - f(x,t'',\xi)| \leq L_1|t' - t''| \quad \forall x \in \overline{\Omega}, t', t'' \in [0, \rho_1[, |\xi| \leq \rho_2 \\ (ii) & |f(x,t,\xi') - f(x,t,\xi'')| \leq L_2|\xi' - \xi''| \quad \forall x \in \overline{\Omega}, t \in [0, \rho_1[, |\xi'|, |\xi''| \leq \rho_2\end{array}$$

where ρ_1 and ρ_2 depend explicitly on $p, N, \theta, a_1, a_2, a_3$ given in the previous hypotheses.

We recall that by a solution of (1.1) we mean a weak solution, that is, a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla \ \ dx = \int_\Omega \frac{f(x, u, \nabla u)}{M(\int_\Omega |\nabla u|^2)} \varphi dx, \quad \text{for all } \varphi \in H_0^1(\Omega).$$

Now, we are in position to establish our main result.

Theorem 1. *Assume hypotheses (H.1)-(H.2) hold. Then ,there exists positive constants* c_1, c_2 *such that for each* $w \in H_0^1(\Omega)$ *then problem* (1.3) *has one solution* u_w *such that*

$$c_1 \leq \|u_w\| \leq c_2 \tag{2.1}$$

where $\|u\| = \left(\int_\Omega |\nabla u|^2\right)^{1/2}$. Moreover, under the above hypotheses, problem (1.3) has a positive and negative solution.

Observation 2.2. *It is well known, that if we are looking only positive solutions, we need assumptions (H.2) (iii)-(iv) only for positive t.*

Theorem 2.*Assume (H.1)-(H.3). Then problem* (1.1) *has a positive and negative solution provided*

$$\frac{4M_2c_2^2 + L_2\lambda_1^{-1/2}}{m_0 - L_1\lambda_1^{-1}} < 1$$

where λ_1 is the first eigenvalue of $-\Delta$ and $M_2 = \text{máx}\{|M'(r)|; 0 \leq r \leq c_2^2\}$. Moreover the solutions obtained are of the class C^2.

3. Proof of theorem 1.

The weak solutions of (1.3) are precisely the critical points of the functional

$$I_w(u) = \frac{1}{2}\|u\|^2 - \int_\Omega H(x, u, \nabla w)dx \tag{3.1}$$

where $H(x, u, \nabla w) = \frac{F(x,u,\nabla w)}{M(\int_\Omega |\nabla w|^2)}$

We will prove, by steps, that I_w has the geometry of the mountain pass theorem, and finally that the obtained solutions have the uniform bounds stated in the theorem.

Step1. Let $w \in H_0^1(\Omega)$. Then there exists positive numbers $\rho, \alpha > 0$ which are independent of w such that

$$I_w(u) \geq \alpha \quad \forall u \in H_0^1(\Omega) : \|u\| = \rho \tag{3.2}$$

Proof. By (H.2)(i). given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|H(x,t,\ )| < \frac{\epsilon t^2}{2m_0} \quad \forall\ |t| \leq \delta$$

and, by (H.2)(ii), there exists $K = K_\delta > 0$ such that

$$|H(x,t,\zeta)| < K|t|^{p+1} \quad \forall\ |t| \geq \delta$$

So, using Sobolev Embedding Theorem ,we get

$$I_w(u) \geq \left(\frac{1}{2} - \frac{\epsilon}{m_0\lambda_1}\right)\|u\|^2 - k_\epsilon\|u\|^{p+1}$$

with k_ϵ a constant independent of w. Since $p > 1$, the thesis easily follows. ■
Step2. Let $w \in H_0^1(\Omega)$. Fix $_0 \in H_0^1(\Omega)$ with $\|\ _0\| = 1$. Then there is a $T > 0$, independent of w, such that

$$I_w(t\ _0) \leq 0 \quad \text{for all} \quad t \geq T \tag{3.3}$$

Proof. First, we observe that, from (H.1)(ii) and (H.2)(iii)

$$|H(x,t,\zeta)| \geq C_1|t|^\theta - C_2, \quad \text{for all} \quad t > t_0 \tag{3.4}$$

So, it follows from (3.4) that

$$\begin{aligned} I_w(t\ _0) &= \frac{1}{2}t^2\|\ _0\|^2) - \int_\Omega H(x,t\ _0,\nabla w)dx \\ &\leq \frac{1}{2}t^2 - c|t|^\theta + C \\ &\to -\infty \quad \text{as} \quad t \to +\infty \end{aligned}$$

due to $\theta > 2$. So, we obtain independent of $_0$ and also w that (3.3) holds.
■
Step3. Let$\{u_n\}$ be a Palais-Smale sequence in $H_0^1(\Omega)$ that is $I_w(u_n) \to c$ and $I_w'(u_n) \to 0$.Then

$$\begin{aligned} c + \|u_n\| &\geq I_w(u_n) - \frac{1}{\theta}\langle I_w'(u_n)\, u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 - \underbrace{\int_\Omega \left(H(x,u_n,\nabla w) - \frac{1}{\theta}\frac{f(x,u_n,\nabla w)}{M(\int_\Omega |\nabla w|^2)}u_n\right)dx}_{L} \end{aligned} \tag{3.5}$$

Here, we claim that L is bounded. Indeed, we consider

$$\Omega_n = \{x \in \Omega : \|u_n(x)\| > T\}$$

with T given in (H.2)(iii), then

$$\begin{aligned} L = &\underbrace{\int_{\Omega_n} \left(H(x,u_n,\nabla w) - \frac{1}{\theta}\frac{f(x,u_n,\nabla w)}{M(\int_\Omega |\nabla w|^2)}u_n\right)dx}_{L_1} + \\ &\underbrace{\int_{\Omega\setminus\Omega_n} \left(H(x,u_n,\nabla w) - \frac{1}{\theta}\frac{f(x,u_n,\nabla w)}{M(\int_\Omega |\nabla w|^2)}u_n\right)dx}_{L_2} \end{aligned} \tag{3.6}$$

But $L_1 \leq 0$ and

$$|L_2| \leq \frac{a_1|\Omega|}{m_0}\left[(1+|T|^p)+\frac{1}{\theta}(|T|+\frac{|T|^{p+1}}{p+1})\right] = K$$

Hence $L \leq K$. So $\{u_n\}$ is bounded in $H_0^1(\Omega)$, and it admits a weakly convergence subsequence. From the Rellich-Kondrachov Theorem and results of weak convergence, standard argument shows that $\{u_n\}$ admits a strongly convergence subsequence.

Step4. From Steps 1-3 and $I_w(0) = 0$, I satisfies the conditions of the mountain pass theorem. So I_w admits at least one nontrivial critical point u_w, at an inf max level,which is a weak solution of (1.3), that is

$$\begin{aligned} &a) I'_w(u_w) = 0 \\ &b) I_w(u_w) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_w(\gamma(t)) \geq \alpha \end{aligned} \tag{3.7}$$

where $\Gamma = \{\gamma \in C^0([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = T\ _0\}$,for some $_0$ and T as in Step 2. From now on we fix such a $_0$ and such a T.

Step5. Let $w \in H_0^1(\Omega)$. There exists a positive constant c_1, independent of w such that

$$\|u_w\| \geq c_1 \tag{3.8}$$

for all solutionu_w obtained in Step 4.

Proof. From the equation (1.3) one gets

$$\int_\Omega |\nabla u_w|^2 dx = \int_\Omega \frac{f(x,u_w,\nabla w)}{M(\int_\Omega |\nabla w|^2)} u_w dx \tag{3.9}$$

By (H.2)(i)-(ii), given $\epsilon > 0$, there exists $c_\epsilon > 0$ independent of w, such that

$$|f(x,t,\nabla w)| \leq \epsilon|t| + c_\epsilon |t|^p$$

So, we get

$$\int_\Omega |\nabla u_w|^2 dx \leq \frac{\epsilon}{m_0}\int_\Omega |u_w|^2 dx + c'_\epsilon \int_\Omega |u_w|^{p+1} dx$$

Hence we have

$$\left(1 - \frac{\epsilon}{\lambda_1 m_0}\right)\|u_w\| \leq \tilde{c}_\epsilon \|u_w\|^{p+1}$$

which implies (3.8) choosing $\epsilon < \lambda_1 m_0$, since $p+1 > 2$ ■

Step6. There exists a positive constant c_2 independent of w such that

$$\|u_w\| \leq c_2$$

Proof. From the infmax characterization of u_w in Step4, choosing the path in Γ as the segment line joining 0 and $_0$, we obtain

$$I_w(u_w) \leq \max_{t\geq 0} I_w(t\ _0)$$

and from (H.3)(iv) we have

$$\begin{aligned} \max_{t\geq 0} I_w(t\ _0) &\leq \max_{t\geq 0}\left\{\frac{t^2}{2}\|\ _0\|^2 - a_2|t|^\theta \int_\Omega |\ _0|^\theta + a_3|\Omega|\right\} \\ &= c_2 \end{aligned} \tag{3.10}$$

Therefore we have obtained that

$$I_w(u_w) \leq c_2$$

Here, using the criticality of u_w for I_w, (3.10), (H.2)(iii), one has

$$\frac{1}{2}\|u_w\|^2 \leq \hat{c}_2 + \frac{1}{\theta}\int_\Omega \frac{f(x,u_w,\nabla w)u_w}{M(\int_\Omega |\nabla w|^2)}$$

Therefore

$$\left(\frac{1}{2} - \frac{1}{m_0\theta}\right)\|u_w\|^2 \leq const.$$

The positivity of u_w it derives from standard arguments. That is one replaces f by $\hat{f}$ defined as

$$\hat{f}(x,t,\xi) = \begin{cases} f(x,t,\xi) & \text{si } t \geq 0 \\ 0 & \text{si } t < 0 \end{cases}$$

Here, we observe that $\hat{f}$ still verifies (H.2)(iii)-(iv)(also we take $v_0 > 0$ in step2). So we find a critical point of mountain-pass type for the corresponding functional $\hat{I_w}$ that is solution of the problem

$$-M(\int_\Omega |\nabla w|^2)\Delta u_w = \hat{f}(x,u_w,\nabla w) \quad \text{in } \Omega,$$
$$u_w = 0 \quad \text{on } \partial\Omega,$$

Multiplying the equation by u_w^- and integrating by parts, we conclude that $u_w^- = 0$. Hence u_w^- is positive. ■

Observation 3.1. *In Step 4 we have obtained a weak solution u_w of (1.3) for each given $w \in H_0^1(\Omega)$.Since $p < \frac{N+2}{N-2}$ a standard bootstrap argument, using regularity theory, shows that $u_w \in C^{0,\alpha}$ for some $\alpha \in (0,1)$. Now, if w have the additional regularity $w \in C^1(\bar{\Omega})$, using the Schauder regularity theory, we show that $u_w \in C^{2,\alpha}(\overline{\Omega})$. As a consequence of the Sobolev embedding theorems and Step6 we conclude that, there exist positive constants ρ_1, ρ_2 such that the solution $u_w$satisfies*

$$\|u_w\|_{C^0} \leq \rho_1, \qquad \|\nabla u_w\|_{C^0} \leq \rho_2$$

4. Proof of theorem 2.

By applying in an iterative way Theorem 1,we construct a sequence $\{u_n\} \subset H_0^1(\Omega)$ whereu_n is a solution of the problem

$$\begin{aligned} -M(\int_\Omega |\nabla u_{n-1}|^2)\Delta u_n &= f(x,u_n,\nabla u_{n-1}) \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

obtained by the mountain pass theorem in theorem 1. We start from as arbitrary $u_0 \in H_0^1(\Omega)\bigcap C^1(\overline{\Omega})$.
By Remark 3.1, we see that

$$\|u_n\|_{C^0} \leq \rho_1, \qquad \|\nabla u_n\|_{C^0} \leq \rho_2$$

Now using (4.1) for u_n we get

$$\begin{aligned} M(\|u_n\|^2)\|u_{n+1} - u_n\|^2 = &\left[M(\|u_{n-1}\|^2) - M(\|u_n\|^2)\right]\int_\Omega \nabla u_n.(\nabla u_{n+1} - \nabla u_n) \\ &+ \int_\Omega [f(x,u_{n+1},\nabla u_n) - f(x,u_n,\nabla u_n)](u_{n+1} - u_n) \\ &+ \int_\Omega [f(x,u_n,\nabla u_n) - f(x,u_n,\nabla u_{n-1})](u_{n+1} - u_n) \end{aligned}$$

hence, from (H.1), (H.3), Cauchy-Schwarz and Poincaré inequalities we have

$$
\begin{aligned}
m_0\|u_{n+1}-u_n\|^2 \leq & 4M_2c_2^2\|u_n-u_{n-1}\|\|u_{n+1}-u_n\| \\
& + L_1\lambda_1^{-1}\|u_{n+1}-u_n\|^2 + L_2\lambda_1^{-1/2}\|u_{n+1}-u_n\|\|u_n-u_{n-1}\|
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
\|u_{n+1}-u_n\| \leq & \frac{4M_2c_2^2 + L_2\lambda_1^{-1/2}}{m_0 - L_1\lambda_1^{-1}}\|u_n-u_{n-1}\| \\
=: & \, k\|u_n-u_{n-1}\|
\end{aligned}
$$

Since the coefficient $k < 1$, we have that $\{u_n\}$ is a Cauchy sequence in H_0^1, and so, $\{u_n\}$ strongly converges in H_0^1 to some function $u \in H_0^1$.
Since $\|u_n\| \geq c_1$, it follows that $u \neq 0$. Hence we find that u is a nontrivial solution of (1.1). By the same argument as in Step6 we have that $u > 0$ in Ω. □

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